Surface Integral

If $\phi(x,y,z) = 0$ is a surface then grad ϕ is along the normal to the surface.

Sums 1. find $\int_S \vec{F} \cdot \vec{n}\, ds$ where $\vec{F} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$ and S is the part of the plane $2x + 3y + 6z = 12$ which is located in the first quadrant.

Soln Here, the surface $\phi(x,y,z) = 0$ is

$$2x + 3y + 6z - 12 = 0$$

$\therefore$ grad ϕ = along the normal to the surface.

Now grad $\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(2x+3y+6z-12)$

$$= 2\vec{i} + 3\vec{j} + 6\vec{k}$$

$\vec{n}$ = unit normal to the surface $2x+3y+6z=12$

$$= \frac{2\vec{i}+3\vec{j}+6\vec{k}}{\sqrt{2^2+3^2+6^2}} = \frac{1}{7}(2\vec{i}+3\vec{j}+6\vec{k})$$

$$\therefore \vec{F}\cdot\vec{n} = (18z\vec{i}-12\vec{j}+3y\vec{k})\cdot\frac{(2\vec{i}+3\vec{j}+6\vec{k})}{7}$$

$$= \frac{1}{7}(2\times 18z - 36 + 18y)$$

$$= \frac{1}{7}[18(y+2z)-36]$$

$$= \frac{1}{7}[6(3y+6z)-36]$$ But $2x+3y+6z=12$

$$\Rightarrow 3y+6z = 12-2x$$

$$\Rightarrow \vec{F}\cdot\vec{n} = \frac{1}{7}[6(12-2x)-36]$$

$$= \frac{1}{7}(36-12x) = \frac{12}{7}(3-x)$$

Now Let S_2 = project of S on xy-plane

$$\Rightarrow \int_S \vec{F}\cdot\vec{n}\,ds = \iint_{S_2} \vec{F}\cdot\vec{n}\,\frac{dx\,dy}{\vec{n}\cdot\vec{K}}$$

$$\Rightarrow \int_S \vec{F}\cdot\vec{n}\,ds = \iint_{S_2} (36-12x)\,dx\,dy$$

$$7x \tfrac{1}{7}(2\vec{i}+3\vec{j}+6\vec{k})\cdot\vec{k}$$

$$= \iint_{S_2} \frac{36-12x}{6}\,dx\,dy$$

$$\Rightarrow \int_S \vec{F}\cdot\vec{n}\,ds = \iint_{S_2} (6-2x)\,dx\,dy$$

In xy-plane, $z=0$.

$\Rightarrow 2x+3y=12 \Rightarrow y = \frac{12-2x}{3}$

So the limit of y will be 0 to $\frac{12-2x}{3}$.

and the limit of x will be 0 to $2x=12$

i.e. 0 to 6.

$$\therefore \int_S \vec{F}\cdot\vec{n}\,ds = \int_{x=0}^{6} \int_{y=0}^{\frac{12-2x}{3}} (6-2x)\,dy\,dx$$

$$\Rightarrow \int_S \vec{F}\cdot\vec{n}\, ds = \int_{x=0}^{6} (6-2x) \left[\int_0^{\frac{12-2x}{3}} dy \right] dx$$

$$= \int_{x=0}^{6} (6-2x) \left[y \right]_0^{\frac{12-2x}{3}} dx$$

$$= \int_{x=0}^{6} (6-2x) \frac{(12-2x)}{3} dx$$

$$= \frac{4}{3} \int_0^6 (x-3)(x-6)\, dx$$

$$= \frac{4}{3} \int_0^6 (x^2 - 9x + 8)\, dx = \frac{4}{3} \left[\frac{x^3}{3} - \frac{9x^2}{2} + 18x \right]$$

$$= \frac{4}{3} \left(72 - 9\times 18 + 108 \right)$$

$$= \frac{4}{3} (180 - 162) = \frac{4}{3} \times 18 = 24$$